Particle acceleration by slow modes in strong compressible MHD turbulence, with application to solar flares.

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ABSTRACT

Energetic particles that undergo strong pitch-angle scattering and diffuse through a plasma containing strong compressible MHD turbulence undergo diffusion in momentum space with diffusion coefficient D_p. If the rms turbulent velocity is of order the Alfvén speed $v_{\rm A}$, the contribution to D_p from slow-mode eddies is $\simeq (2p^2 v_{\rm A}/9l)[\ln(lv_{\rm A}/D_\parallel)+2\gamma-3]$, where l is the outer scale of the turbulence, $\gamma \simeq 0.577$ is Euler's constant, and $D_\parallel$ is the spatial diffusion coefficient of energetic particles, which is assumed to satisfy $D_\parallel \ll lv_{\rm A}$. The energy spectrum of accelerated particles is derived for this value of D_p, taking into account Coulomb losses and particle escape from the acceleration region with an energy-independent escape time.

Slow modes in the $D_\parallel \ll lv_{\rm A}$-limit are an unlikely explanation for electron acceleration in solar flares to energies of 10-100 keV, because for solar-flare conditions the predicted acceleration times are too long and the predicted energy spectra are too hard. The acceleration mechanism discussed in this paper could in principle explain the relatively hard spectra of gyrosynchrotron-emitting electrons in the 100-5000 keV range, but only if $D_\parallel \ll lv_{\rm A}$ for such particles.

1. Introduction

The first part of this paper treats stochastic particle acceleration by slow modes in compressible magnetohydrodynamic (MHD) turbulence assuming efficient pitch-angle scattering. The second part presents the energy spectra of accelerated particles taking into account Coulomb losses and assuming that particles escape the acceleration region on an energy-independent time scale. The third part of the paper shows that slow-modes in the efficient-pitch-angle-scattering limit are not a viable explanation for electron acceleration in solar flares in the 10-100 keV energy range, although in principle they could explain the electron spectra at 100-5000 keV inferred from microwave gyrosynchrotron radiation.

Throughout the paper it is assumed that energetic particles propagate through a magnetized plasma containing both "large-scale" turbulent compressible motions that vary over distances much larger than the energetic-particle gyroradius ρ_g and "small-scale" waves that vary over distances $\sim \rho_g$. The small-scale waves scatter particles, causing diffusion in physical space with mean free path $\lambda_{\rm mfp}$ as well as diffusion in momentum space. The large-scale motions also cause spatial and momentum diffusion. Only those large-scale plasma motions on scales $\gg \lambda_{\rm mfp}$ are considered. Particle acceleration by velocities on scales $\gg \lambda_{\rm mfp}$ has been considered previously by a number of authors [e.g., Kulsrud & Ferrari (1971) Bykov & Toptygin (1982, 1990, 1993), Ptuskin (1988), Dolginov & Silantev 1990, Katz & Stehlik (1991), Chandran & Maron (2003)]. In the present paper, the large-scale velocities are taken to be slow-mode eddies in strong anisotropic compressible MHD turbulence. Such eddies are elongated along the local magnetic field, and involve velocities directed primarily along the local magnetic field (Lithwick & Goldreich 2001, Cho & Lazarian 2002). Part of the motivation for considering slow modes is that under certain conditions they can contribute significantly to momentum diffusion. An additional motivation is for completeness; by comparison the effects of fast modes and Alfvén modes are relatively well understood (e.g., Jokipii 1966, Kulsrud & Pearce 1969, Berezinskii et al 1990, Miller et al 1997, Schlickeiser & Miller 1998, Chandran 2000, Schlickeiser 2002, Yan & Lazarian 2002).

If the small-scale waves are Alfvén waves, the transport equation describing the energetic-particle distribution function f is (Skilling 1975)

$$\frac{\partial f}{\partial t} + \left[\frac{\partial}{\partial p^3}\left(p^3 \mathbf{w}\right)\right]\cdot\nabla f = \nabla\cdot\left(D_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla f\right) + (\nabla\cdot\mathbf{w})\frac{p}{3}\frac{\partial f}{\partial p} + \frac{\partial}{\partial p^3}\left[9p^4 D_{\rm pp}^{\rm (s)}\frac{\partial f}{\partial p^3}\right], \tag{1}$$

where p is momentum, $\hat{\mathbf{b}}$ is the magnetic-field unit vector,

$$\mathbf{w} = \mathbf{u} + \left\langle \frac{3}{2}(1-\mu^2)\frac{\nu_+ - \nu_-}{\nu_+ + \nu_-}\right\rangle v_{\rm A}\hat{\mathbf{b}} \tag{2}$$

is an effective "wave-frame" velocity at which the cosmic rays are advected, $\mathbf{u}$ is the large-scale plasma velocity, $v_{\rm A}$ is the Alfvén speed, ν_+ and ν_- are, respectively, the pitch-angle scattering rates associated with waves traveling parallel to and anti-parallel to the magnetic field, μ is the pitch-angle cosine, $\langle \ldots \rangle$ indicates an average over μ,

$$D_\parallel = v^2 \left\langle \frac{1-\mu^2}{2(\nu_+ + \nu_-)}\right\rangle \tag{3}$$

is the diffusion coefficient for particle motion along the magnetic field,

$$D_{pp}^{\rm (s)} = 4\gamma_L^2 m^2 v_{\rm A}^2 \left\langle \left(\frac{1-\mu^2}{2}\right)\frac{\nu_+\nu_-}{\nu_+ + \nu_-}\right\rangle \tag{4}$$

is the momentum diffusion coefficient associated with the small-scale waves, and γ_L is the relativistic Lorentz factor of an energetic particle. Equations (2) and (4) show that either $\mathbf{w} \neq \mathbf{u}$ or

$D^{\mathrm{s}}_{pp} \neq 0$ (Schlickeiser 2002). Nevertheless, for simplicity the momentum diffusion associated with small-scale waves is ignored ($D^{\mathrm{s}}_{pp} \to 0$), and at the same time, inconsistently, it is assumed that $\mathbf{w} = \mathbf{u}$. This approximation has been standard in studies of particle acceleration by velocities on scales $\gg \lambda_{\mathrm{mfp}}$ [Bykov & Toptygin (1982, 1990, 1993), Ptuskin (1988), Dolginov & Silantev 1990, Katz & Stehlik (1991)]; it reduces equation (1) to

$$\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f = \nabla\cdot\left(D_{\|}\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla f\right) + (\nabla\cdot\mathbf{u})\frac{p}{3}\frac{\partial f}{\partial p}. \tag{5}$$

Equation (5) is used to calculate the momentum diffusion coefficient arising from the large-scale turbulence. It seems plausible that a nonzero D^{s}_{pp} would approximately add to the momentum diffusion coefficient associated with the large-scale turbulence calculated from equation (5), but future work is needed to verify this. The spatial diffusion caused by the large-scale turbulence is neglected. It is assumed that the rms large-scale turbulent velocity is comparable to v_{A} and that the ratio β of thermal to magnetic pressure is $\lesssim 1$. The power spectrum of the slow modes is assumed to follow the theory of Lithwick & Goldreich (2001), which has received support from direct numerical simulations (Cho & Lazarian 2002). It is also assumed that $v_{\mathrm{A}} d_{\mathrm{min}} \ll D_{\|} \ll v_{\mathrm{A}} l$, where d_{min} is the length along the magnetic field of the smallest slow-mode eddies under consideration (d_{min} is either the length of the eddies at the dissipation scale or several times λ_{mfp}, whichever is larger), and l is the length of the largest (outer-scale) eddies. Since the time for an eddy of length $\lambda_{\|}$ (measured along the magnetic field) to be randomized is $\lambda_{\|}/v_{\mathrm{A}}$ (Lithwick & Goldreich 2001), the inequality $v_{\mathrm{A}} d_{\mathrm{min}} \ll D_{\|} \ll v_{\mathrm{A}} l$ implies that the largest eddies are randomized before a particle diffuses through them, and that a particle diffuses through eddies of length d_{min} before such eddies are randomized. The assumption $D_{\|} \gg v_{\mathrm{A}} d_{\mathrm{min}}$ requires that the particle speed v satisfy $v \gg v_{\mathrm{A}}$ since $D_{\|} \sim v\lambda_{\mathrm{mfp}}$ and $d_{\mathrm{min}} > \lambda_{\mathrm{mfp}}$.

The procedure used to calculate the momentum diffusion coefficient D_p arising from the large-scale velocities is to average over the slow-mode fluctuations, neglect spatial variations in the averaged distribution function f_0, and approximate the eddies on scales comparable to l as a uniform background field $\mathbf{B}_0$. The eddies on scales $\ll l$ have velocities $\ll v_{\mathrm{A}}$ and magnetic perturbations $\ll B_0$ and can be treated using quasilinear theory. It is found that

$$\frac{\partial f_0}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D_p \frac{\partial f_0}{\partial p}\right), \tag{6}$$

with

$$D_p \simeq \frac{2p^2 v_{\mathrm{A}}}{9l}\left[\ln\left(\frac{l v_{\mathrm{A}}}{D_{\|}}\right) + 2\gamma - 3\right], \quad \text{for } v_{\mathrm{A}} d_{\mathrm{min}} \ll D_{\|} \ll v_{\mathrm{A}} l, \tag{7}$$

where $\gamma \simeq 0.577$ is Euler's constant. The corresponding acceleration time scale, p^2/D_p, is of order the large-eddy turnover time l/v_{A}, and depends only weakly through the $\ln(l v_{\mathrm{A}}/D_{\|})$ term on p

and particle species. An approximate version of equation (7) is obtained using phenomenological arguments in section 2.3.

Section 3 presents the time-dependent spectrum of accelerated particles assuming an ad-hoc model of particle escape from the acceleration region with an energy-independent escape time. Section 4 gives the steady-state spectrum for electron acceleration taking into account Coulomb losses, again assuming an energy-independent escape time. Section 5 compares the predicted spectra and acceleration times to observations of electrons in solar flares. The results of the paper are summarized in section 6. Much of the notation used in the paper is defined in table 1.

Notation	Meaning
$\lambda_{\rm mfp}$	scattering mean free path of energetic particles
$\lambda_{\rm thermal}$	Coulomb mean free path of thermal protons
$\lambda_{\|}$	length of a turbulent eddy measured along the magnetic field
$\lambda_{\perp}$	width of a turbulent eddy measured across the magnetic field
r_g	energetic-particle gyroradius
f	energetic-particle distribution function
$N(E)$	number of accelerated particles per unit energy
$\mathbf{B}$	magnetic field
$d_{\|}$	length along magnetic field of eddies at the dissipation scale
$d_{\rm min}$	$d_{\|}$ or several times $\lambda_{\rm mfp}$, whichever is larger
l	stirring scale (outer scale) of turbulence
$v_{\rm A}$	Alfvén speed
$c_{\rm s}$	sound speed
β	ratio of thermal to magnetic pressure
$\mathbf{u}$	turbulent velocity
v	energetic-particle velocity
$D_{\|}$	diffusion coefficient for motion along magnetic field
D_p	momentum diffusion coefficient
$\tau_{\rm esc}$	escape time scale
$\tau_{\rm acc}$	acceleration time scale
τ	$t/\tau_{\rm acc}$
χ	$\tau_{\rm acc}/\tau_{\rm esc}$
γ	Euler's constant, ~ 0.577

Table 1: Definitions.

2. Stochastic particle acceleration by slow-mode eddies in strong compressible MHD turbulence

This section reviews properties of small-amplitude slow magnetosonic waves and compressible MHD turbulence and presents phenomenological and analytic derivations of the contribution to D_p from slow modes in compressible MHD turbulence.

2.1. Properties of small-amplitude slow magnetosonic waves

It is instructive to consider the properties of a small-amplitude slow magnetosonic wave propagating in a uniform background magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ with frequency ω, wave vector $\mathbf{k}$, and $k_z \ll k$. Such a wave varies much more rapidly across the magnetic field than along the magnetic field, like slow modes in compressible MHD turbulence (see section 2.2). The wave frequency, polarization, and velocity divergence when $k_z \ll k$ are given by (Lithwick & Goldreich 2001)

$$\omega = \frac{c_s v_A |k_z|}{\sqrt{c_s^2 + v_A^2}} \left[1 + O\left(\frac{k_z}{k}\right)^2\right], \tag{8}$$

$$\frac{u_x}{u_z} \simeq -\frac{c_s^2}{c_s^2 + v_A^2}\,\frac{k_z}{k}, \tag{9}$$

$$u_y = 0, \tag{10}$$

and

$$\mathbf{k}\cdot\mathbf{u} = \frac{v_A^2}{c_s^2 + v_A^2}\left[1 + O\left(\frac{k_z}{k}\right)^2\right] k_z u_z, \tag{11}$$

where $\mathbf{u}$ is the fluctuating velocity, $\mathbf{k}$ is taken to lie in the xz-plane, and c_s is the sound speed. Equations (9) and (10) imply that the slow-wave velocity is approximately aligned with the background magnetic field. The total pressure perturbation (magnetic plus thermal) vanishes to second order in k_z/k (Lithwick & Goldreich 2001). An important property of slow waves with $k_z \ll k$ when $\beta \lesssim 1$ (and thus $c_s \lesssim v_A$) follows from equations (9) and (11): $\mathbf{k}\cdot\mathbf{u} \sim k_z|\mathbf{u}|$.

2.2. Slow-mode eddies in strong anisotropic MHD turbulence

The anisotropy of magnetohydrodynamic (MHD) turbulence has been studied by many authors.[1] In this paper, it is assumed that the ambient plasma is stirred at a scale l, and that the rms turbulent velocity is comparable to v_A. In this case, the rms magnetic fluctuation at scale l, denoted B_l, is comparable to any mean magnetic field in the system. The stirring excites an inertial range of turbulent fluctuations extending from the large scale l to a much smaller dissipation scale. Within any box of dimension $\ll l$, the fluctuations can be decomposed into the three MHD wave polarizations with respect to the average magnetic field direction within the box, $\mathbf{B}_{\rm local}$. For example, when $\beta \ll 1$, the velocity fluctuations aligned with $\mathbf{B}_{\rm local}$ are associated with slow waves. The slow-wave fluctuations within the box can be thought of as a collection of nested eddies, where an eddy is simply a volume of some specified width $\lambda_\perp$ measured across the magnetic field and length $\lambda_\parallel$ measured along the magnetic field, for which the velocity variation across the width of the eddy is comparable to the velocity variation along the length of the eddy. For values of $\lambda_\perp$ in the inertial range, slow-mode eddies are elongated along $\mathbf{B}_{\rm local}$, with

$$\lambda_\parallel \sim \lambda_\perp^{2/3} l^{1/3}, \tag{12}$$

and the rms velocity variation across a slow-mode eddy is

$$u_{\lambda_\perp} \sim v_A \left(\frac{\lambda_\perp}{l}\right)^{1/3} \tag{13}$$

(Lithwick & Goldreich 2001, Cho & Lazarian 2002). Equation (12) means that the velocity varies more rapidly across $\mathbf{B}_{\rm local}$ than along $\mathbf{B}_{\rm local}$. Suppose the vector separation between two points is $\mathbf{r}$, with r in the inertial range of the turbulence, and let the rms velocity difference between the two points be $\delta\mathbf{u}$. If $\mathbf{r} \perp \mathbf{B}_{\rm local}$, equation (13) implies that $\delta u \sim v_A (r/l)^{1/3}$; if $\mathbf{r} \parallel \mathbf{B}_{\rm local}$, equations (12) and (13) imply that $\delta u \sim v_A (r/l)^{1/2}$. Slow-mode eddies are randomized in a time $\sim \lambda_\parallel / v_A$ due to the mixing of slow modes by Alfvén-mode eddies (Lithwich & Goldreich 2001).

The dissipation scale in the directions perpendicular to $\mathbf{B}_{\rm local}$, denoted $d_\perp$, and the corresponding parallel scale $d_\parallel = d_\perp^{2/3} l^{1/3}$, are given by (Lithwick & Goldreich 2001)

$$d_\perp \sim r_{\rm g,\,thermal} \quad \text{for } \beta \ll 1, \text{ and} \tag{14}$$

$$d_\parallel \sim \lambda_{\rm thermal} \quad \text{for } \beta \gtrsim 1, \tag{15}$$

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[1] See, e.g., Montgomery & Turner 1981, Shebalin et al. 1983, Higdon 1984, Higdon 1986, Oughton et al. 1994, Sridhar & Goldreich 1994, Goldreich & Sridhar 1995, Montgomery & Matthaeus 1995, Ghosh & Goldstein 1997, Goldreich & Sridhar 1997, Matthaeus et al. 1998, Spangler 1999, Bhattacharjee & Ng 2000, Cho & Vishniac 2000, Maron & Goldreich 2001, Milano et al 2001, Lithwick & Goldreich 2001, Cho & Lazarian 2002.

where $\lambda_{\rm thermal}$ is the collisional mean free path of thermal ions, and ion-neutral friction and radiative cooling are ignored. Equation (14) is surprising in that linear slow magnetosonic waves are strongly damped in low-β plasmas on scales smaller than the collisional mean free path: i.e., when $\beta \ll 1$, the damping time of slow modes with $\lambda_\| < \lambda_{\rm thermal}$ is $\sim \lambda_\|/c_{\rm s}$, comparable to the linear wave period. The reason that turbulent slow-mode eddies can persist on parallel scales $< \lambda_{\rm thermal}$ when $\beta \sim c_{\rm s}^2/v_{\rm A}^2 \ll 1$ is that the cascade of slow mode eddies is controlled by the Alfvén modes, and the cascade time $\lambda_\|/v_{\rm A}$ is much less than the linear slow-wave damping time (Lithwick & Goldreich 2001).

2.3. Phenomenological estimate of D_p

Since only that part of the turbulence with $\lambda_\| \gg \lambda_{\rm mfp}$ is considered, and since the momentum diffusion associated with small-scale waves is ignored, the time derivative of a particle's momentum induced by the large-scale velocity **u** is given by (Ptuskin 1988)

$$\frac{dp}{dt} = -\frac{\nabla\cdot u}{3}\ p. \tag{16}$$

For $\beta \lesssim 1$, the rms velocity divergence of slow-mode eddies of width $\lambda_\perp$ satisfies (Lithwick & Goldreich 2001)

$$\langle|\nabla\cdot\mathbf{u}|^2_{\lambda_\perp}\rangle^{1/2} \sim \frac{u_{\lambda_\perp}}{\lambda_\|}. \tag{17}$$

The rms contribution to dp/dt from eddies of length $\lambda_\|$ is thus

$$\left(\frac{dp}{dt}\right)_{\lambda_\|} \sim \frac{p v_{\rm A}}{3\sqrt{\lambda_\| l}}. \tag{18}$$

If a particle interacts coherently with an eddy of length $\lambda_\|$ for a time $\triangle t$, it incurs a random momentum increment of rms magnitude $\triangle p \sim (dp/dt)\triangle t$. The contribution to D_p from eddies of size $\lambda_\|$ is $\sim (\triangle p)^2/\triangle t$. When $d_{\rm min} v_{\rm A} \ll D_\| \ll l v_{\rm A}$, particles are confined within eddies with $\lambda_\| > D_\|/v_{\rm A}$ throughout the time $\lambda_\|/v_{\rm A}$ required for the eddies to be randomized in the turbulent flow. For such eddies, $\triangle t \sim \lambda_\|/v_{\rm A}$. Each eddy size between $D_\|/v_{\rm A}$ and l makes a contribution to D_p of $\sim p^2 v_A/(9l)$. The contribution to D_p from all such eddies is

$$D_p \sim \frac{p^2 v_{\rm A}}{9l}\ \ln\left(\frac{l v_{\rm A}}{D_\|}\right) \qquad \text{for } d_{\rm min} \ll (D_\|/v_{\rm A}) \ll l, \tag{19}$$

in approximate agreement with equation (30) below. The contribution to D_p from eddies with $\lambda_\parallel < D_\parallel/v_{\rm A}$ can be neglected. [2]

2.4. Derivation of D_p in the quasilinear approximation.

Particle acceleration by slow-mode eddies on scales $\ll l$ is now treated analytically for $\beta \lesssim 1$. A scale l' is introduced with $\lambda_{\rm mfp} \ll l' \ll l$, and an approximation is made in which eddies with $l' < \lambda_\perp < l$ are replaced with a uniform magnetic field $B_l\hat{\mathbf{z}}$, so that $\mathbf{B}_{\rm local}$ is everywhere approximately along $\hat{\mathbf{z}}$. In terms of the Fourier transform of the velocity $\mathbf{u}(\mathbf{k},t)$, and in accord with equations (12) and (13), the spectrum of the (homogeneous) turbulence is taken to be (Goldreich & Sridhar 1995, Lithwick & Goldreich 2001, Cho & Lazarian 2002)

$$\langle \mathbf{u}(\mathbf{k}_1,t_1)\cdot\mathbf{u}(\mathbf{k}_2,t_2)\rangle = P(\mathbf{k}_1)e^{-\gamma_k|t_1-t_2|}\delta(\mathbf{k}_1+\mathbf{k}_2), \tag{20}$$

with

$$P(\mathbf{k}) = \frac{v_{\rm A}^2}{6\pi}\, l^{-1/3}k_\perp^{-10/3} g\left(\frac{k_z\, l^{1/3}}{k_\perp^{2/3}}\right) \tag{21}$$

for $(l')^{-1} < k_\perp < d_\perp^{-1}$ [$P(\mathbf{k}) = 0$ otherwise],

$$g(x) = e^{-|x|}, \tag{22}$$

and

$$\gamma_k = k_\perp^{2/3} l^{-1/3} v_{\rm A} \sim \frac{v_{\rm A}}{\lambda_\parallel}, \tag{23}$$

where $k_\perp = \sqrt{k_x^2+k_y^2}$. Given equation (17), it is assumed that

$$\langle \mathbf{k}_1\cdot\mathbf{u}(\mathbf{k}_1,t_1)\;\; \mathbf{k}_2\cdot\mathbf{u}(\mathbf{k}_2,t_2)\rangle$$
$$= -k_{1z}^2\, P(\mathbf{k}_1)e^{-\gamma_{k1}|t_1-t_2|}\delta(\mathbf{k}_1+\mathbf{k}_2). \tag{24}$$

[2] If slow-mode eddies were oscillatory, as opposed to repeatedly randomized, then eddies larger than $D_\parallel/v_{\rm A}$ would contribute less to D_p than eddies of length $D_\parallel/v_{\rm A}$, and the logarithmic term would disappear from equation (19). However, although $\nabla\cdot\mathbf{u}$ is oscillatory within any infinitesimal fluid element (since the density does not change secularly in time), the average value of $\nabla\cdot\mathbf{u}$ over the volume of a slow-mode eddy is randomized in the time $\lambda_\parallel/v_{\rm A}$ since the eddy is completely shredded and mixed with its neighbors in a time $\sim\lambda_\parallel/v_{\rm A}$ by Alfvén modes. In addition, an energetic particle moves from one fluid element within an eddy of length $\lambda_\parallel$ to a new fluid element that is chosen randomly from the surrounding eddy-volume of width $\lambda_\perp$ and length $\lambda_\parallel$ in a time $\lambda_\parallel/v_{\rm A}$ due to cross-field diffusion and the turbulent mixing of fluid elements by Alfvén modes.

For $\mathbf{B}_{\rm local} \propto \hat{\mathbf{z}}$, equation (5) becomes

$$\frac{\partial f}{\partial t} + \mathbf{u}\cdot\nabla f = D_\parallel \frac{\partial^2 f}{\partial z^2} + (\nabla\cdot\mathbf{u})\frac{p}{3}\frac{\partial f}{\partial p}. \tag{25}$$

The distribution function is written as the sum of two parts,

$$f = f_0 + f_1, \tag{26}$$

where $f_0 = \langle f\rangle$, $f_1 = f - \langle f\rangle$, and angled brackets denote an average over an ensemble of realizations of the turbulence. For simplicity, it is assumed that $\langle u\rangle = 0$ and $\nabla f_0 = 0$. The ensemble average of equation (25) is then

$$\frac{\partial f_0}{\partial t} = -\langle \mathbf{u}\cdot\nabla f_1\rangle + \left\langle (\nabla\cdot\mathbf{u})\frac{p}{3}\frac{\partial f_1}{\partial p}\right\rangle. \tag{27}$$

Subtracting equation (27) from equation (25) yields

$$\frac{\partial f_1}{\partial t} - D_\parallel \frac{\partial^2 f_1}{\partial z^2} = \frac{(\nabla\cdot\mathbf{u})p}{3}\frac{\partial f_0}{\partial p}, \tag{28}$$

in which products of fluctuating quantities have been neglected. Such products are small compared to $|\partial f_1/\partial t|$ since the slow-mode velocity is almost along z, $u \ll v_{\rm A}$ for eddies on scales $\ll l$, and $|\partial f_1/\partial t| \sim v_{\rm A}|\partial f_1/\partial z|$. Upon solving for f_1 and substituting into equation (27), one recovers equation (6) with

$$D_p = \frac{p^2}{9}\int d^3k \int_0^\infty d\tau\ k_z^2 P(\mathbf{k}) e^{-(\gamma_k + k_z^2 D_\parallel)\tau}. \tag{29}$$

When $d_{\rm min} \ll (D_\parallel/v_{\rm A}) \ll l'_\parallel$,

$$D_p = \frac{2p^2 v_{\rm A}}{9l}\left[\ln\left(\frac{l'_\parallel v_{\rm A}}{D_\parallel}\right) + 2\gamma - 3\right], \tag{30}$$

where $\gamma \simeq 0.577$ is Euler's constant. An estimate of the contribution to D_p from all eddies, including those with $l'_\parallel < \lambda_\parallel < l$, is obtained by taking $l'_\parallel \to l$ in equation (30).

3. Time-dependent energy spectrum of accelerated particles assuming an energy-independent escape time and no Coulomb losses

In this section, the energy spectrum of accelerated particles is presented for $p > p_0$, where p_0 corresponds to a particle velocity of a few times $v_{\rm A}$. Coulomb collisions are neglected, and particle

escape from the acceleration region is modeled in an ad hoc fashion by adding a loss term to the right-hand side of equation (6),

$$\frac{\partial f}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D_p \frac{\partial f}{\partial p}\right) - \frac{f}{\tau_{\rm esc}}, \tag{31}$$

where the escape time $\tau_{\rm esc}$ is assumed to be momentum-independent. In equation (31) and throughout this section, the subscript on f_0 will be dropped, with the understanding that f now refers to the distribution function obtained after averaging over the turbulent fluctuations. Equation (31) is solved for $p \geq p_0$ and $t \geq 0$ subject to the boundary conditions

$$f(p = p_0,\, t \geq 0) \;=\; A \quad \text{with } A \text{ constant}, \tag{32}$$

$$f(p > p_0,\, t = 0) \;=\; 0, \quad \text{and} \tag{33}$$

$$f(p \to \infty,\, t \geq 0) \;\to\; 0. \tag{34}$$

Equation (32) would apply, at least approximately, if Coulomb collisions kept the distribution at low energies fixed and only a small minority of the particles were accelerated. It is assumed that

$$D_p = \frac{p^2}{\tau_{\rm acc}}, \tag{35}$$

with $\tau_{\rm acc}$ independent of p. From equation (30) it can be seen that $\tau_{\rm acc}$ for acceleration by slow-mode eddies in compressible MHD turbulence is of order $l/v_{\rm A}$ and independent of p and particle species, aside from a possible weak momentum and species dependence arising from the $\ln(l_\parallel^\prime v_{\rm A}/D_\parallel)$ term. The solution to equations (31) through (34) is (Kardashev 1962, Schlickeiser 2002)[3]

$$f = \frac{A}{2}\left(\frac{p}{p_0}\right)^{-(3+\sqrt{9+4\chi})/2}\left[1 + {\rm erf}\left(\frac{-\ln(p/p_0) + \tau\sqrt{9+4\chi}}{2\sqrt{\tau}}\right)\right]$$
$$+\frac{A}{2}\left(\frac{p}{p_0}\right)^{(-3+\sqrt{9+4\chi})/2}\left[1 - {\rm erf}\left(\frac{\ln(p/p_0) + \tau\sqrt{9+4\chi}}{2\sqrt{\tau}}\right)\right], \tag{36}$$

where

$$\tau \equiv \frac{t}{\tau_{\rm acc}}, \tag{37}$$

$$\chi = \frac{\tau_{\rm acc}}{\tau_{\rm esc}}, \tag{38}$$

and

$$ {\rm erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy \tag{39}$$

is the error function.

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[3]See also Parker (1957), Ramaty (1979), Schlickeiser (1984), Ball et al (1992), and Park & Petrosian (1995).

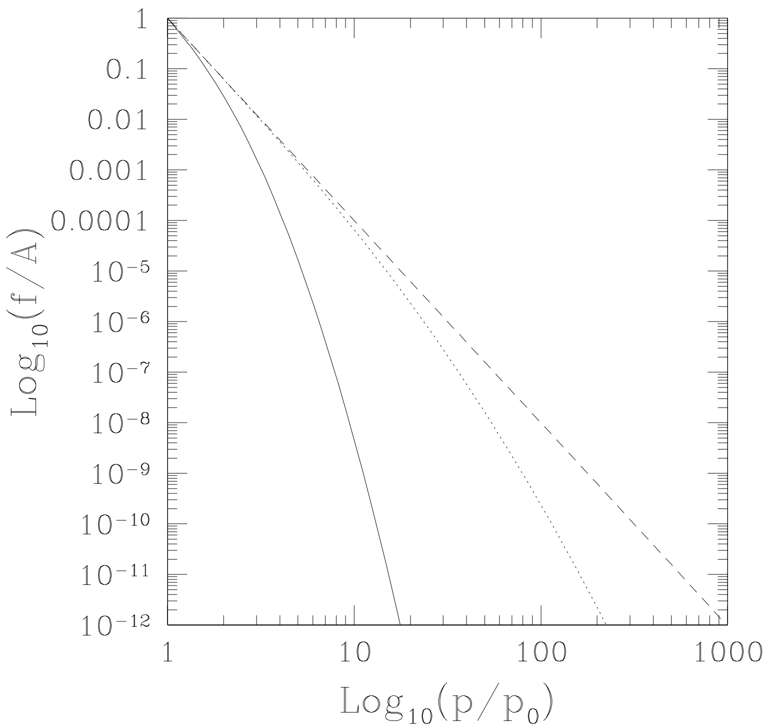

Fig. 1.— The distribution function f at $t = \tau_{\rm acc}/10$ (solid line), $t = \tau_{\rm acc}/2$ (dotted line), and $t = 2\tau_{\rm acc}$ (dashed line) for $\tau_{\rm acc} = 4\tau_{\rm esc}$, when Coulomb losses are neglected.

The solution in equation (36) is illustrated in figure 1 for $\chi = 4$. The solid line corresponds to $\tau = 0.1$, the dotted line to $\tau = 0.5$, and the dashed line to $\tau = 2$. Roughly speaking, f approaches the power law $A(p/p_0)^{-(3+\sqrt{9+4\chi})/2}$ for $p = p_0$ up to $p = p_{\rm max} \sim p_0 e^{\tau\sqrt{9+4\chi}}$, with f dropping off sharply for $p > p_{\rm max}$. The width of the cutoff in $\ln(p/p_0)$-space increases like $\sqrt{\tau}$. For the case $\chi = 4$ plotted in figure 1, $f \sim p^{-4}$ for $p \lesssim p_{\rm max} = p_0 e^{5\tau}$. When $\tau_{\rm acc} \gg \tau_{\rm esc}$, $p_{\rm max} \sim e^{2t/\sqrt{\tau_{\rm acc}\tau_{\rm esc}}}$, and the de-facto time scale for particle acceleration is $\sim \sqrt{\tau_{\rm esc}\tau_{\rm acc}}$. This time scale is short for small $\tau_{\rm esc}$ because the only particles to reach high energies when $\tau_{\rm esc} \ll \tau_{\rm acc}$ are a small fraction that happen to be accelerated much more rapidly than average.

The energy spectrum $N(E)$ is the number of accelerated particles per unit energy. In the non-relativistic limit, it is given by

$$N(E) = 4\pi m p f(p)\Big|_{p=\sqrt{2mE}} \quad , \tag{40}$$

where m is the particle mass. In the non-relativistic limit, $N(E)$ approaches the power law E^{-q}

for $E < p_{\rm max}^2/2m$, with $q = (1+\sqrt{9+4\chi})/4$; equivalently, a specified power law $N(E) \propto E^{-q}$ corresponds to $\chi = 4q^2 - 2q - 2$. In the ultra-relativistic limit, $N(E)$ approaches the power law E^{-q} for $E < p_{\rm max}c$, with $q = (-1+\sqrt{9+4\chi})/4$; equivalently, for a specified q, $\chi = 4q^2 + 2q - 2$. If escape is neglected, then $\chi = 0$; this corresponds to $q = 1$ in the non-relativistic limit and $q = 1/2$ in the ultra-relativistic limit.

4. Steady-state energy spectrum of accelerated electrons with Coulomb losses and an energy-independent escape time

In the presence of Coulomb losses, steady electron injection at energy E_0, and an energy-independent escape time $\tau_{\rm esc}$, the energy spectrum of superthermal electrons $N(E)$ satisfies the Fokker-Planck equation (Park & Petrosian 1995)

$$\frac{\partial N}{\partial t} = \frac{\partial^2}{\partial E^2}(D_E N) - \frac{\partial}{\partial E}(A_E N) - \frac{N}{\tau_{\rm esc}} + S\delta(E-E_0)\Theta(t), \tag{41}$$

where in the non-relativistic limit

$$D_E = \frac{p^2 D_p}{m_e^2}, \tag{42}$$

$$A_E = \frac{m_e}{p^2}\frac{d}{dp}(pD_E) - \frac{\alpha_c m_e^2 c^3}{p}, \tag{43}$$

$$\alpha_c = 6\times 10^{-3}\left(\frac{n}{10^{10}\ {\rm cm}^{-3}}\right)\ {\rm s}^{-1}, \tag{44}$$

n is the electron density, S is a constant, and $\Theta(t)$ is the Heavyside function. Assuming

$$D_p = \frac{p^2}{\tau_{\rm acc}}, \tag{45}$$

with $\tau_{\rm acc}$ independent of p, the steady-state ($t \to \infty$) solution to equation (41) for $E > E_0$ is [Park & Petrosian 1995, equation (71)][4]

$$N(E) = c_1 x^{3\tilde{\gamma}/2} M(\tilde{a}, \tilde{b}, |\beta| x^{-3/2}), \tag{46}$$

where c_1 is a constant,

$$x = \frac{E}{m_e c^2}, \tag{47}$$

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[4]Equation (46) of this paper follows from the results of Park & Petrosian (1995) upon noting two minor typos in that paper. In particular, the second line of equation (67) of that paper should read $\delta_\pm = [(a+1)/2] \pm \mu$, while the second half of equation (68) of that paper should read $\lambda_0 = [(a+1)/2]^2 + \theta$ (Petrosian 2003). The correct versions of the equations were given by Park (1995).

$$\tilde{a} = \frac{1}{2} + \frac{\sqrt{9+4\chi}}{6}, \tag{48}$$

$$\tilde{b} = 1 + \frac{\sqrt{9+4\chi}}{3}, \tag{49}$$

$$\tilde{\gamma} = -\frac{1}{6} - \frac{\sqrt{9+4\chi}}{6}, \tag{50}$$

and

$$\beta = \frac{\alpha_c \tau_{\rm acc}}{6\sqrt{2}}. \tag{51}$$

Equation (46) is illustrated in figure 2 for the parameters $E_0 = 0.002\, m_e {\rm c}^2$, $n = 2 \times 10^9$ cm^{-3}, $\chi = 10$, and $\tau_{\rm acc} = 5$ s. For $x \gg \beta^{2/3}$, Coulomb losses become unimportant, and the spectrum obtains the same steady-state power law as in section 3. For $x \lesssim \beta^{2/3}$, Coulomb losses cause the spectrum to steepen relative to its high-energy power law (Hamilton & Petrosian 1992).

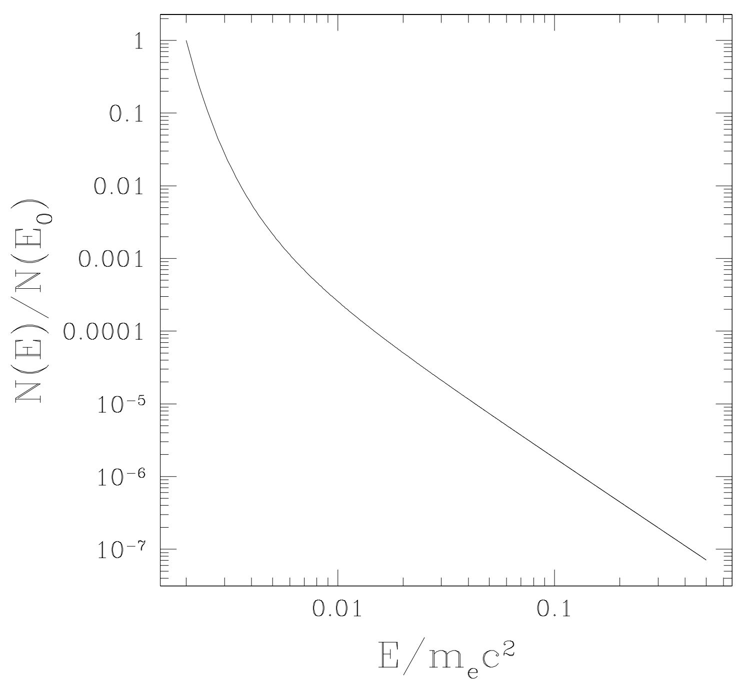

Fig. 2.— The steady-state energy spectrum of accelerated electrons taking into account Coulomb losses and an energy-independent escape time, assuming $E_0 = 0.002\, m_e {\rm c}^2$, $n = 2 \times 10^9$ cm^{-3}, $\chi = 10$, and $\tau_{\rm acc} = 5$ s.

5. Application to electron acceleration in solar flares

In this section it is shown that the acceleration mechanism discussed in this paper is probably not responsible for electron acceleration in solar flares in the 10-100 keV range. On the other hand, the predicted energy spectrum is roughly consistent with the energy spectra of electrons in the 100-5000 keV range, although it is not clear that $D_{\parallel} \ll l v_{\rm A}$ for such particles.

A simple and fairly standard flare model is considered, as depicted in figure 3, with features drawn from the works of, e.g., Sturrock (1966), LaRosa et al (1994), Tsuneta (1996), Miller et al (1996), Tsuneta et al (1997), Miller et al (1997), and Aschwanden (2002). The flare arises from reconnection between open magnetic field lines above a pair of footpoints. The reconnection layer is assumed to be Petschek-like, so that the reconnection rate is sufficiently fast. Most of the magnetic energy is released outside (downstream) of the reconnection layer, where the magnetic tension associated with newly reconnected field lines accelerates plasma away from the reconnection site in a "sling-shot" action. It is assumed that turbulence is generated in this magnetized outflow with rms velocity $\sim v_{\rm A}$, and that the turbulence is similar to strong homogeneous MHD turbulence (section 2.2). Electrons accelerated by the turbulence stream towards the solar surface, evaporating chromospheric plasma and emitting hard bremsstrahlung x-rays at the pair of footpoints that anchor the magnetic flux tube occupied by the energetic electrons. The evaporated chromospheric plasma rises, filling a magnetic-flux-tube loop with hot relatively dense plasma that emits soft x-rays. As additional field lines reconnect, the hard x-ray footpoints move progressively outward and the top of the soft-x-ray loop rises. Time-of-flight measurements typically place the electron acceleration region well above the tops of the soft x-ray loops and also above the hard x-ray emission that has been observed above the soft x-ray loops in a number of flares (Aschwanden 2002).

The electron energy spectrum varies between flares and between the footpoints and above-the-looptop regions. If $N(E)$ is taken to be a power-law E^{-q} in the 10-100 keV range, then a typical value of q is 4, although values between 1.5 and ~ 10 have been observed (Alexander & Metcalf 1997, Miller et al 1997, Petrosian 2002). Above ~ 100 keV, the electron spectrum flattens, with values of q in the range $2.0-2.7$ for the three flares modeled by Kundu et al (2001). Time-of-flight measurements indicate that the electron acceleration time is very short. For the flare of December 15, 1991, 1932 UT, Aschwanden (2002) found that the time to energize electrons to 511 keV is $\ll 0.36$ s; Aschwanden (2002) pointed out that even the 0.18 s acceleration time to 511 keV found by Miller et al (1996) is too long to be consistent with the energy-dependent arrival times of electrons at the hard-x-ray footpoints in this flare.

The start frequencies of bi-directional and type III radio bursts indicate that the electron density in the acceleration region is in the range $n_e = 0.6 - 10 \times 10^9$ cm^{-3} (Aschwanden & Benz 1997). Models of the gyrosynchrotron emission suggest that the magnetic field strength at the tops of flaring loops is ~ 300 G (Kundu et al 2001). The magnetic field in the acceleration region

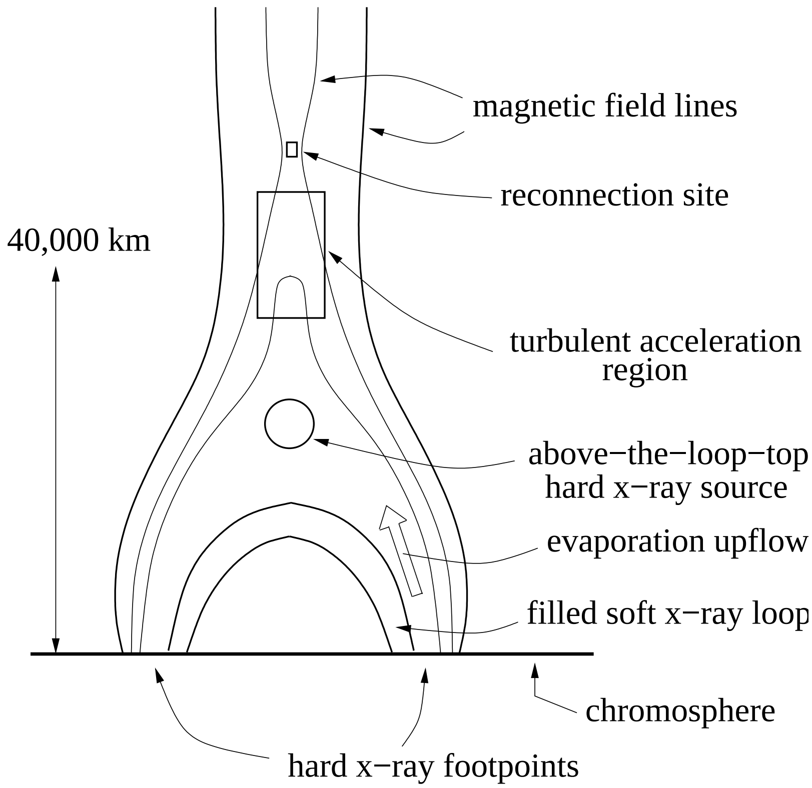

Fig. 3.— Reconnection model for a solar flare with a single pair of footpoints.

is probably somewhat smaller. For typical parameters, it is assumed that $n_e = 2 \times 10^9$ cm^{-3} and $B = 200$ G in the acceleration region, yielding $v_{\rm A} \sim 10^4$ km/s. It is assumed that the outer scale of the turbulence is $l = 5000$ km, and that the region of greatest turbulence intensity is of size $\sim l$.

It has been hypothesized that microturbulence in the acceleration region, e.g., whistler turbulence, causes $\lambda_{\rm mfp}$ for energetic electrons to be much shorter than the Coulomb mean free path (Miller et al 1997). If $D_\parallel \ll l v_{\rm A}$ as assumed in this paper, then the time for particles to diffuse out of the region of strongest turbulence, $l^2/D_\parallel$, is much longer than the acceleration time, $\tau_{\rm acc} \sim l/v_{\rm A}$. On the other hand, the plasma undergoes a bulk flow at speed $\sim v_{\rm A}$ away from the reconnection site that advects electrons out of the region of strongest turbulence on a time scale $l/v_{\rm A}$. For particles with $D_\parallel \ll l v_{\rm A}$, transport out of the turbulent region is dominated by this advection, and the escape time $\tau_{\rm esc}$ is then $\sim l/v_{\rm A} \sim 0.5$ s, independent of electron energy (Blackman 2003). If $d_{\rm min} v_{\rm A} \ll D_\parallel \ll l v_{\rm A}$, then particle acceleration by slow-modes is approximately described by equation (30). Taking $l_\parallel^\prime \rightarrow l$ in equation (30) and $D_\parallel = l v_{\rm A}/10$, one finds that $\tau_{\rm acc} = p^2/D_p \sim 10 l/v_{\rm A} \sim 5$ s, and $\chi = \tau_{\rm acc}/\tau_{\rm esc} = 10$.

Coulomb losses are the dominant energy losses for electrons in solar flares with $E < 100$ keV (Hamilton & Petrosian 1992). If escape is modeled with an ad-hoc loss term as in equation (41), then the energy spectrum of electrons accelerated by slow modes approaches the form depicted in figure 2. For energies above a few keV, Coulomb losses can be neglected and $N(E) \propto E^{-q}$, with $q = 2$. Such a spectrum is significantly harder than typical solar-flare spectra in the 10-100 keV range, as discussed above. For energies above a few keV, the time dependent spectrum behaves approximately as in section 3, with $N(E) \propto E^{-2}$ for $E < E_{\rm max}$, with $E_{\rm max} = p_{\rm max}^2/2m \propto e^{(2t/\tau_{\rm acc})\sqrt{9+4\chi}} = e^{14t/\tau_{\rm acc}}$. The time required for $E_{\rm max}$ to increase from 5 keV to 500 keV is then $\sim 0.3\tau_{\rm acc} \sim 1.5$ s. This time is more than a factor of 10 larger than the energization time implied by time-of-flight measurements for the flare of December 15, 1991, 1932 UT (Aschwanden 2002). The inability of the acceleration mechanism described in this paper to explain the observed steep electron spectra and short acceleration times makes it an unlikely explanation for electron acceleration in solar flares in the 10-100 keV range.

The predicted spectra are roughly consistent with the energy spectra of electrons in the 100-5000 keV range inferred from microwave gyrosynchrotron emission (Kundu et al 2001). The electrons responsible for this emission may have considerably longer acceleration times than the electrons involved in x-ray bursts. The acceleration mechanism discussed in this paper could explain these high-energy electrons, but only if $D_\parallel \ll l v_{\rm A}$ for electrons in this energy range.

6. Summary

The momentum diffusion coefficient D_p arising from slow modes in strong compressible anisotropic MHD turbulence is calculated for the case that the rms turbulent velocity is $\sim v_{\rm A}$, pitch-angle scattering of energetic particles is very efficient ($d_{\rm min} v_{\rm A} \ll D_\parallel \ll l v_{\rm A}$), the energetic particle speed v is $\gg v_{\rm A}$, and $\beta \lesssim 1$. It is found that

$$D_p \simeq \frac{2p^2 v_{\rm A}}{9l}\left[\ln\left(\frac{l v_{\rm A}}{D_\parallel}\right) + 2\gamma - 3\right], \qquad \text{for } v_{\rm A} d_{\rm min} \ll D_\parallel \ll v_{\rm A} l, \tag{52}$$

where $\gamma \simeq 0.577$ is Euler's constant. Aside from a possible weak momentum dependence associated with the $\ln(l v_{\rm A}/D_\parallel)$ term, $D_p \propto p^2$, implying that the acceleration time $\tau_{\rm acc} = p^2/D_p$ is independent of p. In addition, $\tau_{\rm acc}$ is of order $l/v_{\rm A}$ and approximately independent of particle species.

Slow modes in the $D_\parallel \ll l v_{\rm A}$-limit are an unlikely explanation for electron acceleration in solar flares to energies of 10-100 keV, because for solar-flare conditions the predicted acceleration times are too long and the predicted energy spectra are too hard. The acceleration mechanism

discussed in this paper could in principle explain the relatively hard spectra of gyrosynchrotron-emitting electrons in the 100-5000 keV range, but only if $D_{||} \ll l v_{\rm A}$ for such particles.

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